



# Displacer gap losses in beta and gamma Stirling engines



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## ABSTRACT

An analytical model has been developed to evaluate "displacer gap losses" in the clearance between the displacer and the cylinder in both  $\beta$  and  $\gamma$  configurations of Stirling engines. Displacer gap losses are the sum of the "shuttle heat transfer" and the "enthalpy pumping".

The present model takes into account the pressure gradient in the gap, the gas compressibility and real gas effect. Gas velocity and temperature distributions in the displacer, in the gap and in the cylinder were determined by solving momentum and energy balances in a concentric tubes geometry. Our model is then introduced in a whole engine model and the effect of the clearance thickness and engine's speed on total displacer gap losses are investigated. Novel tendencies of the solution are observed and new ways for optimization are demonstrated.

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## 1. Introduction

The Stirling engine has been invented since the 19th century but was unable to penetrate the energy and transport sector despite its flexibility and relatively high thermal efficiency. The use of this technology has been concentrated in cryogenics and spatial applications. However, with the present growth of concerns about energy saving and environmental issues, Stirling engines becomes more interesting than ever. It is the best solution to make micro-scale co-generation since the ORC (organic Rankine cycle) is relatively less efficient for a power level below 100 kW<sub>el</sub> [1]. Stirling cycle implies that a constant mass of working gas (usually air, helium or hydrogen) alternates between two temperature levels and passes through four thermodynamic processes: an isothermal compression, an isochoric heating, an isothermal expansion and finally an isochoric cooling [2]. In  $\beta$  and  $\gamma$  configurations of Stirling engines (Fig. 1) this reciprocating motion is assured by a displacer. To avoid frictional losses and excessive wear, clearance seals are generally used instead of ring seals. The inconvenience of this solution is that a gas leakage occurs from the compression to the expansion chamber or back depending on the displacer's direction of motion and the pressure difference between these spaces. This

leakage causes an enthalpy pumping through the displacer's clearance [3].

Besides, a temperature gradient exists along the cylinder wall. Thus, the displacer is moving between a hot and a cold region and causes heat transfer with its motion. It takes some heat from the hot region of the cylinder and rejects it in the cold region. This amount of lost heat is called "shuttle heat transfer". The sum of enthalpy pumping and shuttle heat transfer loss are known in the literature as "Dispalcer gap losses" [3].

Urieli [4], used a steady state model (steady displacer velocity and axial pressure gradient) to calculate the leakage mass flow through the displacer's clearance. Urieli's formula was used for several years. Rios [5] developed an approximate solution by linearization and application of Fourier series to calculate the shuttle heat transfer. Later, Baik and Chang [6] developed an analytic solution for shuttle heat transfer by considering only conduction in both cylinder and displacer and by neglecting the effect of the gap flow between them. This effect was introduced later by Chang et al. [7]. But, they considered a simple geometry (parallel plates) and assumed that the gas flow in the gap is only due to the displacer motion without any effect of the pressure gradient in the gap. Finally, Kotsubo and Swift [8] applied the thermoacoustic theory to study enthalpy pumping and shuttle heat transfer. They considered a parallel plates geometry and supposed that the leakage flow is *a priori* known.

Some numerical studies exist also like the 1D study proposed by Andersen [9] and the work of Huang and Berggren [10]. These studies have been conducted for specific engines and used a

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| Nomenclature         |                                                                        | $\lambda$           | thermal conductivity, $\text{W K}^{-1} \text{m}^{-1}$ |
|----------------------|------------------------------------------------------------------------|---------------------|-------------------------------------------------------|
| $C_p$                | specific heat capacity, $\text{J K}^{-1} \text{kg}^{-1}$               | $\nu$               | kinematic viscosity, $\text{m}^2 \text{s}^{-1}$       |
| $L$                  | length, m                                                              | $\omega$            | angular speed, $\text{rad s}^{-1}$                    |
| $p$                  | pressure, Pa                                                           | <b>Subscripts</b>   |                                                       |
| $Pr$                 | Prandtl number                                                         | 0                   | time-average                                          |
| $R$                  | radius, m or perfect gas spec const., $\text{J kg}^{-1} \text{K}^{-1}$ | $c$                 | cylinder or compression space                         |
| $r$                  | radial coordinate, m                                                   | $d$                 | displacer                                             |
| $T$                  | temperature, K                                                         | $e$                 | expansion space                                       |
| $t$                  | time, s                                                                | $f$                 | fluid                                                 |
| $u$                  | velocity, $\text{ms}^{-1}$                                             | $H$                 | high temperature source                               |
| $V$                  | volume, $\text{m}^3$                                                   | $h$                 | hot HEX                                               |
| $x$                  | position, m                                                            | $k$                 | cold HEX                                              |
| $z$                  | axial coordinate, m                                                    | $L$                 | low temperature sink                                  |
| <b>Greek letters</b> |                                                                        | $n$                 | $n$ th order of Fourier series                        |
| $\alpha$             | thermal diffusivity, $\text{m}^2 \text{s}^{-1}$                        | <b>Superscripts</b> |                                                       |
| $\beta$              | isochoric thermal pressure coefficient, $\text{K}^{-1}$                | *                   | complex number                                        |
| $\Gamma$             | temperature gradient in the gap, $\text{Km}^{-1}$                      | $\sim$              | oscillating part                                      |
| $\gamma$             | ratio $C_p/C_v$                                                        |                     |                                                       |

convective heat transfer coefficient calculated from correlations. Besides, numeric studies are greedy in terms of computing time and then can not be used in whole engine optimization approaches.

In the present study, efforts are put in modeling analytically displacer gap losses by solving momentum and energy equations in the annular gap between the cylinder and the displacer. Unlike Baik [6] and Chang [7], flow and pressure gradient in the gap are taken into account here and that affects considerably solutions and new phenomena are observed. Introducing our model in a whole engine model, we demonstrate the existence of an optimal clearance thickness and an optimal engine speed that minimizes displacer gap losses.

## 2. Problem formulation

### 2.1. Problem description

As shown in Fig. 1, The displacer separates the compression and the expansion spaces. The pressure difference between these spaces is caused by the friction of the gas flowing through the heat exchangers and the regenerator. The problem is modeled in this work as a pulsating flow in the annular gap between two concentric tubes: the displacer and the cylinder. The displacer is moving periodically relative to the cylinder. In Fig. 2, two types of coordinate systems are used: a stationary system ( $z, r$ ) for the cylinder and the gap fluid and a moving system ( $z_d, r_d$ ) for the displacer. These coordinate systems are related by the displacer's motion:

$$r_d = r \quad (1a)$$

$$z_d = z - x_d(t) \quad (1b)$$

where  $x_d(t)$  is the axial coordinate of the origin of the moving coordinate system in the stationary one. It represents also the position of the displacer relative to the cylinder.

### 2.2. Assumptions

As said previously, we keep some assumptions from previous works [6–8]. These are:

1. The fluid flow is laminar and fully developed.
2. The thickness of the cylinder or the displacer is greater than the thermal penetration depth.
3. Radiation heat transfer is neglected.
4. Axial temperature gradients in the displacer, in the cylinder and in the gap are constant and equal to each other. This gradient is approximated by  $\Gamma = T_H - T_L/L_d$ .

However,

- Contrary to Baik [6] and Chang [7], the gas is not assumed incompressible and pressure gradient effect in the gap is taken

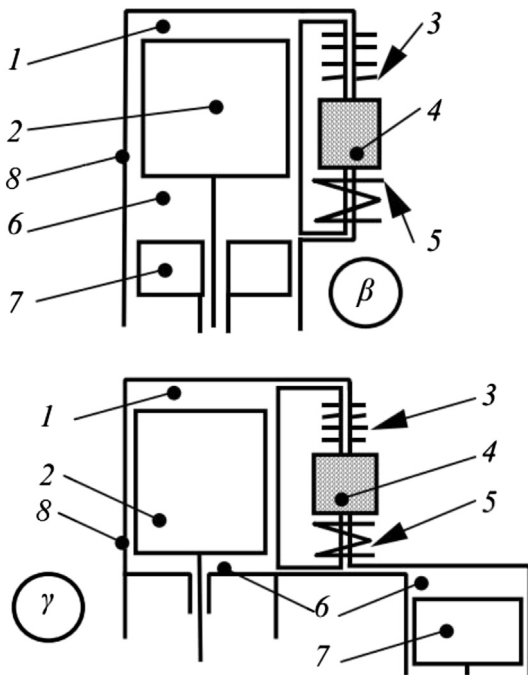


Fig. 1.  $\beta$  and  $\gamma$  configurations of the Stirling engine: 1 – expansion chamber; 2 – displacer; 3 – heater; 4 – regenerator; 5 – cooler; 6 – compression chamber; 7 – power piston; 8 – cylinder.

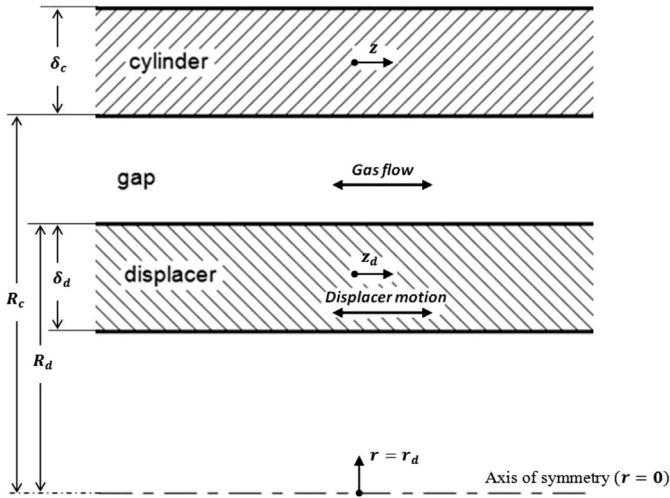


Fig. 2. Problem's geometry and coordinate systems used.

into account but pressure variations in compression and expansion spaces are considered small compared to their mean value. This is generally the case for Stirling engines to avoid Hysteresis losses.

- contrary to Kotsubo [8] the leakage flow rate is not considered known.
- contrary to all of them we choose to use cylindrical coordinates instead of parallel plates geometrie.
- contrary to all of them, we dont use a first order approximation to model fluctuations and this allows us to introduce our model into a realistic engine model.

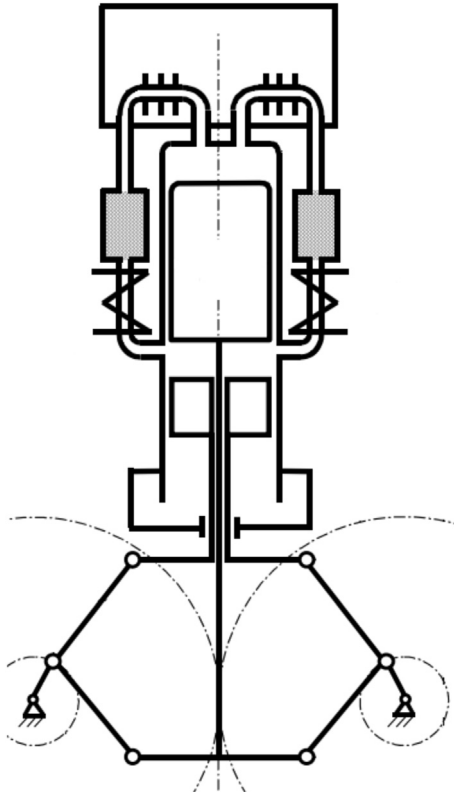


Fig. 3. GPU-3 engine (β type with Rhombic mechanism).

In a Stirling engine gas velocity and displacer velocity are fluctuating with zero mean value.  $x_d(t)$  has also a zero mean value in our coordinate system (Fig. 2). However, gas pressure and different temperatures are composed of a time averaged and time-dependent parts.

Respectively to previous assumptions, we have:

- In the fluid gap

$$p(z, t) = p_0(z) + \tilde{p}(z, t) \quad (2a)$$

$$u(z, r, t) = \tilde{u}(z, r, t) \quad (2b)$$

$$T_f(z, r, t) = T_{\text{ref}} - \Gamma z + \tilde{T}_f(r, t) \quad (2c)$$

- In the displacer

$$u_d(t) = \tilde{u}_d(t) \quad (3a)$$

$$x_d(t) = \tilde{x}_d(t) \quad (3b)$$

$$\begin{aligned} T_d(z_d, r_d, t) &= T_{\text{ref}} - \Gamma z_d + \tilde{T}_d(r_d, t) \\ &= T_{\text{ref}} - \Gamma z + \Gamma \tilde{x}_d(t) + \tilde{T}_d(r, t) \end{aligned} \quad (3c)$$

- In the cylinder

$$T_c(z, r, t) = T_{\text{ref}} - \Gamma z + \tilde{T}_c(r, t) \quad (4)$$

where  $T_{\text{ref}}$  is a reference temperature, which is simply selected as the cylinder temperature at a location with  $z = 0$ .

### 2.3. Governing equations

According to previews assumptions, momentum and energy equations for the flow in the gap are given in the literature [11]:

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{1}{\rho_{f0}} \frac{\partial \tilde{p}}{\partial z} + \nu_{f0} \left( \frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{u}}{\partial r} \right) \quad (5)$$

$$\frac{\partial \tilde{T}_f}{\partial t} + \frac{\partial T_{f0}}{\partial z} \tilde{u} - \frac{T_{f0} \beta_0}{\rho_{f0} C_{pf0}} \frac{\partial p}{\partial t} = \alpha_{f0} \left( \frac{\partial^2 \tilde{T}_f}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{T}_f}{\partial r} \right) \quad (6)$$

In order to take into account heat conduction in the cylinder and in the displacer, we write governing equations in these parts:

$$\frac{\partial T_c}{\partial t} = \alpha_{c0} \left( \frac{\partial^2 T_c}{\partial r^2} + \frac{1}{r} \frac{\partial T_c}{\partial r} \right) \quad (7a)$$

$$\frac{\partial T_d}{\partial t} = \alpha_{d0} \left( \frac{\partial^2 T_d}{\partial r^2} + \frac{1}{r} \frac{\partial T_d}{\partial r} \right) \quad (7b)$$

### 2.4. Boundary conditions

We use non-slip boundary conditions for gas velocity:

$$u(R_d, t) = u_d(t) \quad (8a)$$

$$u(R_c, t) = 0 \quad (8b)$$

According to assumption 3, the cylinder and the displacer can be considered as semi-infinite walls. Thus, temperatures at  $r = R_d - \delta_d$

**Table 1**  
Geometric parameters of the GPU-3 engine.

| Parameter                  | Value                  |
|----------------------------|------------------------|
| <i>Clearance volumes</i>   |                        |
| Compression space          | 28.68 cm <sup>3</sup>  |
| Expansion space            | 30.52 cm <sup>3</sup>  |
| <i>Swept volumes</i>       |                        |
| Compression space          | 113.14 cm <sup>3</sup> |
| Expansion space            | 120.82 cm <sup>3</sup> |
| <i>Heater</i>              |                        |
| Tube number                | 40                     |
| Tube inside diameter       | 3.02 mm                |
| Tube length                | 245.3 mm               |
| Void volume                | 70.88 cm <sup>2</sup>  |
| <i>Cooler</i>              |                        |
| Tube number/cylinder       | 312                    |
| Tube inside diameter       | 1.08 mm                |
| Length of the tube         | 46.1 mm                |
| Void volume                | 13.8 cm <sup>3</sup>   |
| <i>Regenerator</i>         |                        |
| Diameter                   | 22.6 mm                |
| Length                     | 22.6 mm                |
| Wire diameter              | 40 μm                  |
| Porosity                   | 0.697                  |
| Unit number/cylinder       | 8                      |
| Thermal conductivity       | 15 W/mK                |
| Void volume                | 50.55 cm <sup>3</sup>  |
| <i>Displacer's length</i>  | 7.01 cm                |
| <i>Cylinder's diameter</i> | 6.96 cm                |

and  $r = R_c + \delta_c$  are steady and not affected by the perturbation caused by the fluid flow. Temperatures at these radius are linear:

$$T_c(z, R_c + \delta_c, t) = T_{ref} - I z \quad (9a)$$

$$T_d(z_d, R_d - \delta_d, t) = T_{ref} - I z_d \quad (9b)$$

Eqs. (9a) and (9b) allow us to write boundary conditions at  $r = R_d - \delta_d$  and  $r = R_c + \delta_c$ :

$$\tilde{T}_c(R_d + \delta_c, t) = 0 \quad (10a)$$

$$\tilde{T}_d(R_c - \delta_d, t) = 0 \quad (10b)$$

Finally, we have a temperature and heat flux continuity conditions at  $r = R_c$  and  $r = R_d$ :

$$T_c(z, R_c, t) = T_f(z, R_c, t) \quad (11a)$$

$$T_d(z, R_d, t) = T_f(z, R_d, t) \quad (11b)$$

$$\lambda_{d0} \frac{\partial T_d(z, R_d)}{\partial r} = \lambda_{f0} \frac{\partial T_f(z, R_d)}{\partial r} \quad (11c)$$

$$\lambda_{f0} \frac{\partial T_f(z, R_c)}{\partial r} = \lambda_{c0} \frac{\partial T_c(z, R_c)}{\partial r} \quad (11d)$$

**Table 2**  
Operating conditions of the considered engine.

| Parameter          | Value     |
|--------------------|-----------|
| Working gas        | Helium    |
| Frequency          | 41.7 Hz   |
| Hot temperature    | 977 K     |
| Cold temperature   | 288 K     |
| Mean pressure      | 4.13 M Pa |
| Experimental power | 3.9 kW    |

### 3. Analytic solutions for governing equations

In order to solve governing equations in a general case we use Fourier series. In fact, all variables of the problem are periodic and can be expressed as Fourier series [12]. When taking complex notation, we can write:

$$p^*(z, t) = p_0(z) + \sum_{n=1}^{\infty} p_n^*(z) e^{i\omega t} \quad (12a)$$

$$u^*(z, r, t) = \sum_{n=1}^{\infty} u_n^*(z, r) e^{i\omega t} \quad (12b)$$

$$u_d^* = \sum_{n=1}^{\infty} u_{dn}^* e^{i\omega t} \quad (12c)$$

$$x_d^* = \sum_{n=1}^{\infty} x_{dn}^* e^{i\omega t} \quad (12d)$$

$$T_f^*(z, r, t) = T_{f0}(z) + \sum_{n=1}^{\infty} T_{fn}^*(r) e^{i\omega t} \quad (12e)$$

$$T_c^*(z, r, t) = T_{c0}(z) + \sum_{n=1}^{\infty} T_{cn}^*(r) e^{i\omega t} \quad (12f)$$

$$T_d^*(z, r, t) = T_{d0}(z) + \sum_{n=1}^{\infty} T_{dn}^*(r) e^{i\omega t} \quad (12g)$$

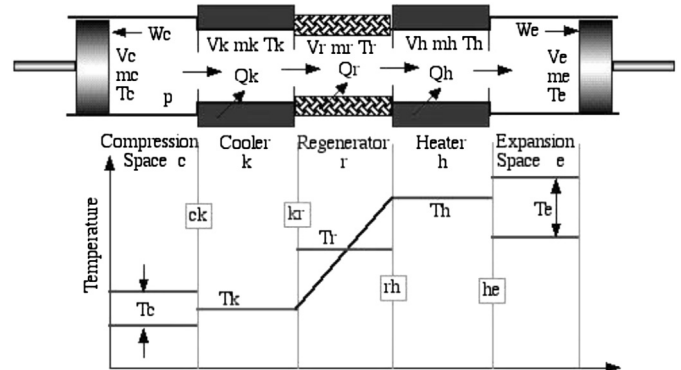
#### 3.1. Velocity distribution of the gap fluid

By reformulating Eq. (5) in complex notation and regrouping terms of the same order. We have for the  $n$ th order:

$$\frac{\partial^2 u_n^*}{\partial r^2} + \frac{1}{r} \frac{\partial u_n^*}{\partial r} - \frac{i n \omega}{\nu_{f0}} u_n^* = \frac{1}{\rho_{f0} \nu_{f0}} \frac{\partial p_n^*}{\partial z} \quad (13)$$

Equation (13) is a modified Bessel equation. Then, it can be solved analytically. Its solution is [13]:

$$u_n^* = -\frac{1}{i \rho_{f0} n \omega} \frac{\partial p_n^*}{\partial z} + C_n I_0(\varepsilon_n r) + D_n J_0(\varepsilon_n r) \quad (14)$$



**Fig. 4.** Schematic model for five compartments of the Stirling engine and temperature distribution in each compartment as in Ref. [4].

**Table 3**

Set of differential and algebraic equations for the ideal Adiabatic Stirling engine model.

| Type                     | Equation                                                                                                                                                                                                                                         |
|--------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Pressure                 | $dp = -\gamma p(dV_c/T_{ck} + dV_e/T_{he})/V_c/T_{ck} + \gamma(V_k/T_k + V_r/T_r + V_h/T_h + V_e/T_{he})$                                                                                                                                        |
| Masses                   | $m_k = pV_k/RT_k$<br>$m_r = pV_r/RT_r$<br>$m_h = pV_h/RT_h$<br>$m_e = M - (m_c + m_k + m_h + m_e)$                                                                                                                                               |
| Temperatures             | $T_c = pV_c/Rm_c$<br>$T_e = pV_e/Rm_e$                                                                                                                                                                                                           |
| Mass variations          | $dm_c = (pDV_c + V_cDp/\gamma)/RT_{ck}$<br>$dm_k = m_k dp/p$<br>$dm_r = m_r dp/p$<br>$dm_h = m_h dp/p$                                                                                                                                           |
| Mass flow rates          | $\dot{m}_{ck} = -dm_c$<br>$\dot{m}_{kr} = \dot{m}_{ck} - dm_k$<br>$\dot{m}_{rh} = \dot{m}_{kr} - dm_r$<br>$\dot{m}_{he} = \dot{m}_{rh} - dm_e$                                                                                                   |
| Conditional temperatures | $T_{ck} = T_c$ if $\dot{m}_{ck} > 0$<br>$T_{ck} = T_k$ if $\dot{m}_{ck} < 0$<br>$T_{he} = T_h$ if $\dot{m}_{he} > 0$<br>$T_{he} = T_e$ if $\dot{m}_{he} < 0$                                                                                     |
| Energy                   | $dW = p(dV_c + dV_e)$<br>$dQ_k = V_k dp_{cv}/R - c_p(T_{ck}\dot{m}_{ck} - T_{kr}\dot{m}_{kr})$<br>$dQ_r = V_r dp_{cv}/R - c_p(T_{kr}\dot{m}_{kr} - T_{rh}\dot{m}_{rh})$<br>$dQ_h = V_h dp_{cv}/R - c_p(T_{rh}\dot{m}_{rh} - T_{he}\dot{m}_{he})$ |

where  $I_0$  et  $J_0$  are respectively modified Bessel functions of the first and second species and  $\varepsilon_n = \sqrt{in\omega/\nu_{f0}}$ .

Constants  $C_n$  and  $D_n$  are determined using non-slip boundary conditions:

$$u_n^*(R_d, t) = u_{dn}^* \quad (15a)$$

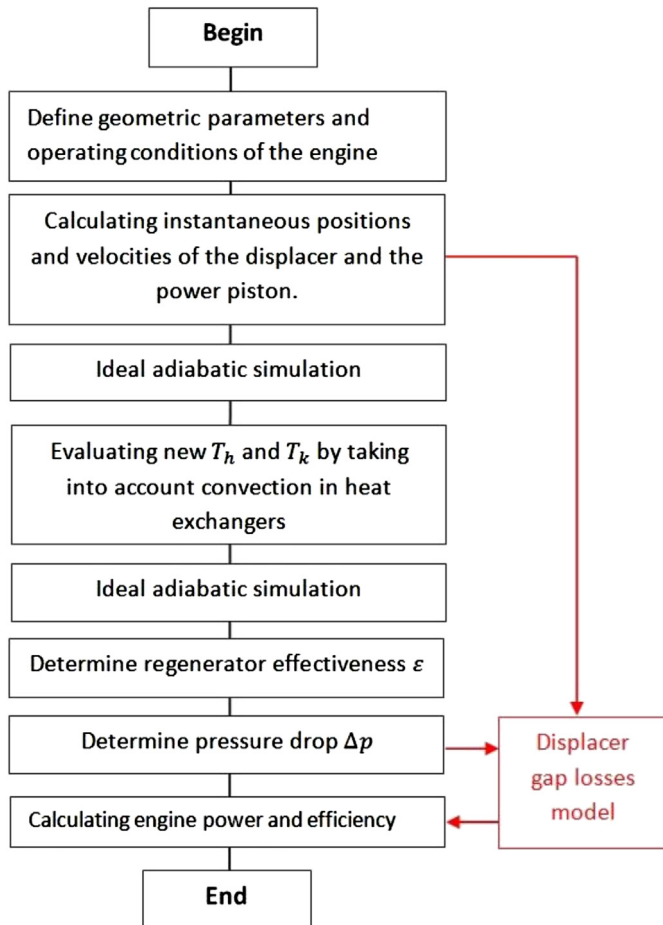
$$u_n^*(R_c, t) = 0 \quad (15b)$$

Expressions of these constants are:

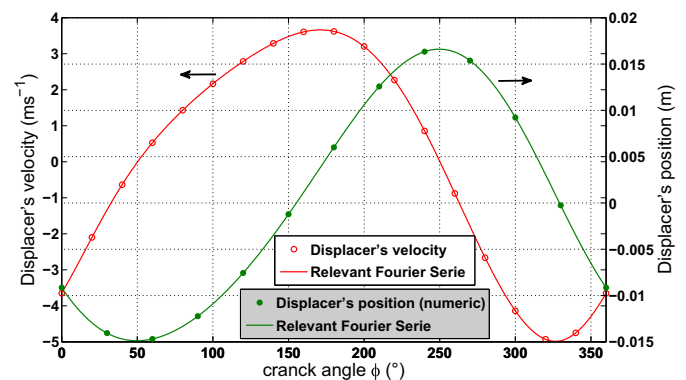
$$C_n = \frac{\frac{1}{i\rho_{f0}n\omega} \frac{\partial p_n^*}{\partial z} [J_0(\varepsilon_n R_c) - J_0(\varepsilon_n R_d)] + u_{dn}^* J_0(\varepsilon_n R_c)}{I_0(\varepsilon_n R_d) J_0(\varepsilon_n R_c) - J_0(\varepsilon_n R_d) I_0(\varepsilon_n R_c)} \quad (16a)$$

$$D_n = \frac{\frac{1}{i\rho_{f0}n\omega} \frac{\partial p_n^*}{\partial z} [I_0(\varepsilon_n R_d) - I_0(\varepsilon_n R_c)]}{I_0(\varepsilon_n R_d) J_0(\varepsilon_n R_c) - J_0(\varepsilon_n R_d) I_0(\varepsilon_n R_c)} + \frac{u_{dn}^*}{J_0(\varepsilon_n R_d) - J_0(\varepsilon_n R_c)} \times \left[ 1 - \frac{J_0(\varepsilon_n R_c) [I_0(\varepsilon_n R_d) - I_0(\varepsilon_n R_c)]}{I_0(\varepsilon_n R_d) J_0(\varepsilon_n R_c) - J_0(\varepsilon_n R_d) I_0(\varepsilon_n R_c)} \right] \quad (16b)$$

Both  $C_n$  and  $D_n$  have two parts: the first part is relative to pressure gradient fluctuations in the gap and the second is relative



**Fig. 5.** Chart flow of the simple analysis and introduction of the displacer gap losses model.



**Fig. 6.** Position and velocity of the displacer.

to the displacer's velocity. Gas velocity fluctuation is the consequence of these two causes. The real velocity is derived from the complex one using the following relation:

$$u(t) = \sum_{n=1}^{\infty} \Re(u_n^*) \cos(n\omega t) - \sum_{n=1}^{\infty} \Im(u_n^*) \sin(n\omega t) \quad (17)$$

### 3.2. Expression of leakage mass flow rate

Leakage mass flow can be calculated by integrating the product of the density and the velocity over the area of the gap. In complex notation, it comes:

$$\dot{m}_l^*(t) = 2\pi\rho_{f0} \int_{R_d}^{R_c} u^*(r, t) r dr \quad (18)$$

We give an analytical expression of this complex mass flow rate:

$$\begin{aligned} \dot{m}_{ln}^* = & -\frac{\pi}{i\omega} \frac{\partial p_n^*}{\partial z} (R_c^2 - R_d^2) + \frac{2\pi\rho_{f0}C_n}{\varepsilon_n} [R_c I_1(\varepsilon_n R_c) - R_d I_1(\varepsilon_n R_d)] \\ & - \frac{2\pi\rho_{f0}D_n}{\varepsilon_n} [R_c J_1(\varepsilon_n R_c) - R_d J_1(\varepsilon_n R_d)] \end{aligned} \quad (19)$$

The real leakage mass flow rate is then given by the following expression:

$$\dot{m}_l(t) = \sum_{n=1}^{\infty} \Re(\dot{m}_{ln}^*) \cos(n\omega t) - \sum_{n=1}^{\infty} \Im(\dot{m}_{ln}^*) \sin(n\omega t) \quad (20)$$

### 3.3. Temperature distribution

By reformulating Eqs. (6), (7a) and (7b) in complex notations and regrouping terms of the same order  $n$ , we obtain:

$$\frac{\partial^2 T_{fn}^*}{\partial r^2} + \frac{1}{r} \frac{\partial T_{fn}^*}{\partial r} - \varepsilon_n^2 Pr T_{fn}^* = \frac{-i}{n\omega} \frac{\partial T_{f0}}{\partial z} \varepsilon_n^2 Pr u_n^* - \varepsilon_n^2 Pr \frac{T_{f0}\beta_0}{\rho_{f0}C_{pf0}} p_n^* \quad (21a)$$

$$\frac{\partial^2 T_{dn}^*}{\partial r^2} + \frac{1}{r} \frac{\partial T_{dn}^*}{\partial r} - \varepsilon_n^2 Pr \sigma_d T_{dn}^* = 0 \quad (21b)$$

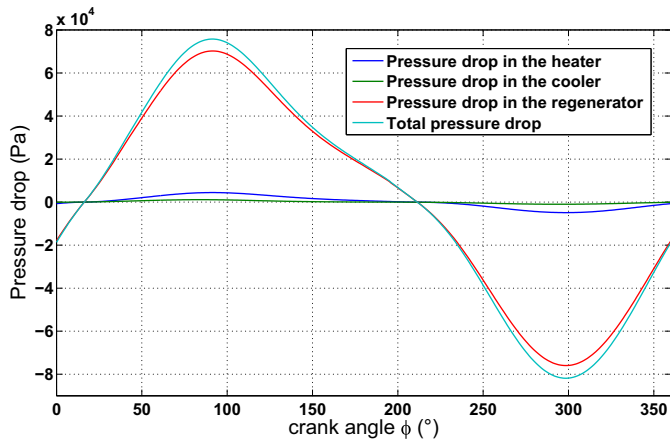


Fig. 7. Pressure drops in the heater, the regenerator and the couler and the total pressure drop  $p_c - p_e$ .

$$\frac{\partial^2 T_{cn}^*}{\partial r^2} + \frac{1}{r} \frac{\partial T_{cn}^*}{\partial r} - \varepsilon_n^2 Pr \sigma_c T_{cn}^* = 0 \quad (21c)$$

where  $Pr$  is the gas Prandtl number,  $\sigma_c = \alpha_{f0}/\alpha_{c0}$  and  $\sigma_d = \alpha_{f0}/\alpha_{d0}$ .

Solutions of Eqs. (21a), (21b) and (21c) are:

$$\begin{aligned} T_{fn}^* = & -\frac{\Gamma}{\rho_{f0}(n\omega)^2} + i \frac{Pr}{Pr-1} \frac{\Gamma C_n}{n\omega} I_0(\varepsilon_n r) + i \frac{Pr}{Pr-1} \frac{\Gamma D_n}{n\omega} J_0(\varepsilon_n r) \\ & + E_n I_0(\varepsilon_n \sqrt{Pr} r) + F_n J_0(\varepsilon_n \sqrt{Pr} r) + \frac{T_{f0}\beta_0 p_n^*}{\rho_{f0}C_{pf0}} \end{aligned} \quad (22a)$$

$$T_{dn}^* = K_n I_0(\varepsilon_n \sqrt{Pr\sigma_d} r) + L_n J_0(\varepsilon_n \sqrt{Pr\sigma_d} r) \quad (22b)$$

$$T_{cn}^* = G_n I_0(\varepsilon_n \sqrt{Pr\sigma_c} r) + H_n J_0(\varepsilon_n \sqrt{Pr\sigma_c} r) \quad (22c)$$

where  $E_n, F_n, G_n, H_n, K_n$ , and  $L_n$  are integration constants which can be calculated using boundary conditions given by Equations (9a) and (9b)–(11d). Writing these equations using Fourier series we have a system of equations for each order  $n$ :

$$T_{dn}^*(R_d - \delta_d) = 0 \quad (23a)$$

$$\frac{\partial T_{dn}^*(R_d)}{\partial r} = \frac{\lambda_{f0}}{\lambda_{d0}} \frac{\partial T_{fn}^*(R_d)}{\partial r} \quad (23b)$$

$$T_{dn}^*(R_d) + \Gamma x_{dn}^* = T_{fn}^*(R_d) \quad (23c)$$

$$\frac{\partial T_{cn}^*(R_c)}{\partial r} = \frac{\lambda_{f0}}{\lambda_{c0}} \frac{\partial T_{fn}^*(R_c)}{\partial r} \quad (23d)$$

$$T_{cn}^*(R_c) = T_{fn}^*(R_c) \quad (23e)$$

$$T_{cn}^*(R_c + \delta_c) = 0 \quad (23f)$$

This equations system is linear and can be solved by hand, but solutions obtained are lengthy. Thus, we choose to solve it numerically.

### 3.4. Calculating displacer gap losses

At an arbitrary axial position  $z$ , the instantaneous local enthalpy flow rate from the high to the low temperature side in the gap and the displacer are given by Ref. [8]:

- In the gap

$$\dot{H}_g(t) = 2\pi \int_{R_d}^{R_c} (\rho_{f0} C_{pf0} T_f u + (1 - T_{f0}\beta_0) p u) r dr \quad (24)$$

- In the displacer

$$\dot{H}_d(t) = 2\pi \int_{R_d - \delta_d}^{R_d} (\rho_{d0} C_{pd0} T_d u_d) r dr \quad (25)$$

Notice that real gas effects are taken into account by the term  $(1 - T_{f0}\beta_0)p u$  which is null for ideal gas.



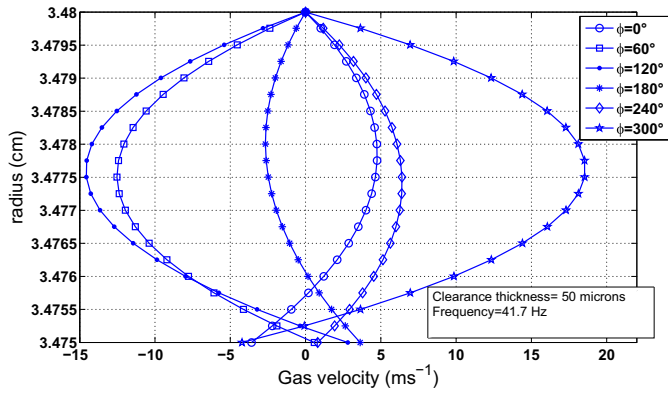


Fig. 8. Gas velocity distribution at different crank angles.

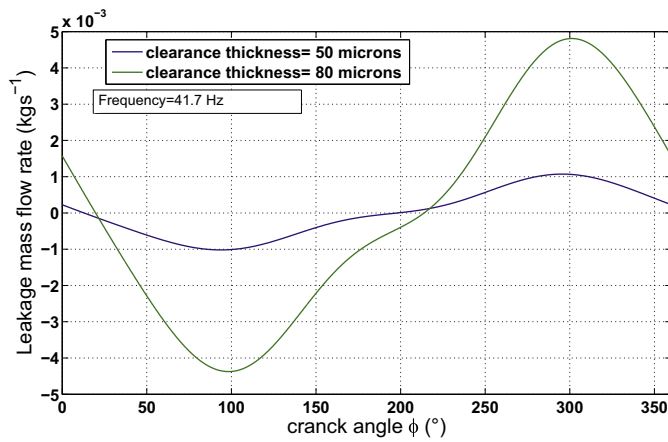


Fig. 9. Leakage mass flow through the displacer's gap.

Displacer gap losses are the sum of "enthalpy pumping" through the displacer's clearance and "shuttle heat transfer". "The enthalpy pumping" is defined as the cycle averaged net enthalpy flow rate from the high to the low temperature side in the fluid gap. The "shuttle heat transfer" is defined as the cycle averaged net enthalpy flow rate in the displacer. Then, displacer gap losses are given by Refs. [6–8]:

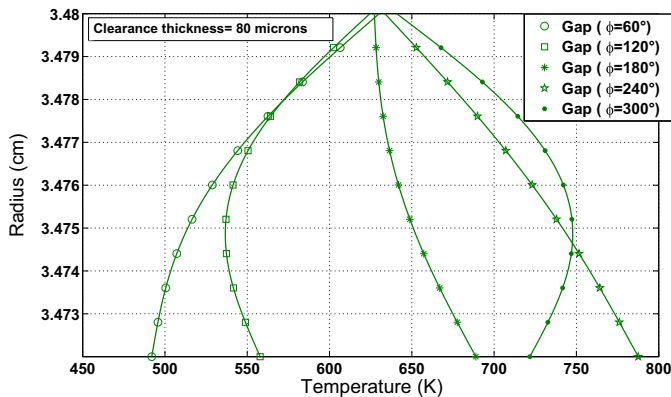


Fig. 10. Radial distribution of gas temperature in the gap at  $z = 0$  and at different crank angles.

$$\dot{H}_l(t) = \frac{\omega}{2\pi} \int_0^{2\pi} \dot{H}_g(t) dt + \frac{\omega}{2\pi} \int_0^{2\pi} \dot{H}_d(t) dt \quad (26)$$

#### 4. Case study

In order to calculate leakage losses in a real Stirling engine, the GPU-3 engine [14] is chosen as a case study. This  $\beta$  type Stirling engine was developed by "General Motors" in 1965. It is shown in Fig. 3 and its specifications and operating conditions are gathered in Tables 1 and 2. Prior to evaluate its displacer gap losses, we have to develop a thermal model for the engine.

##### 4.1. Thermal model of the engine

The simple analysis is used in order to model the engine. This analysis is a decoupled losses approach based on the ideal adiabatic model proposed by Urieli and Brechowitz [4]. Stirling engine is divided, as illustrated in Fig. 4, on five compartments: an expansion space, a heater, a regenerator, a cooler and a compression space. There are four interfaces between them through which the working gas passes from a compartment to another. Governing equations of the adiabatic model are obtained by applying energy and mass balances and the equation of state on each compartment. The set of equations of the ideal adiabatic model are gathered in Table 3. It contains non-linear differential equations to be integrated numerically. We use a classical fourth order Runge-Kutta method for the

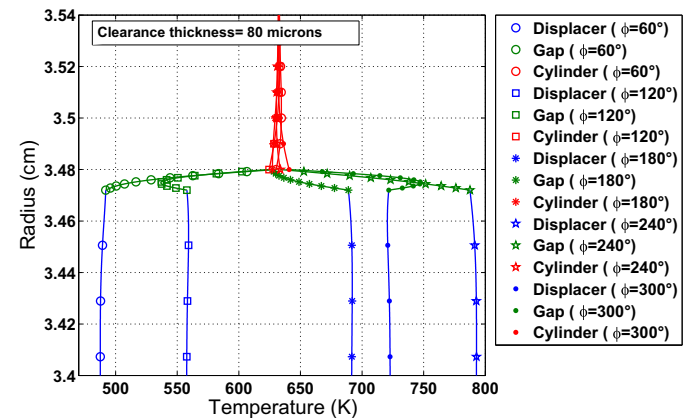
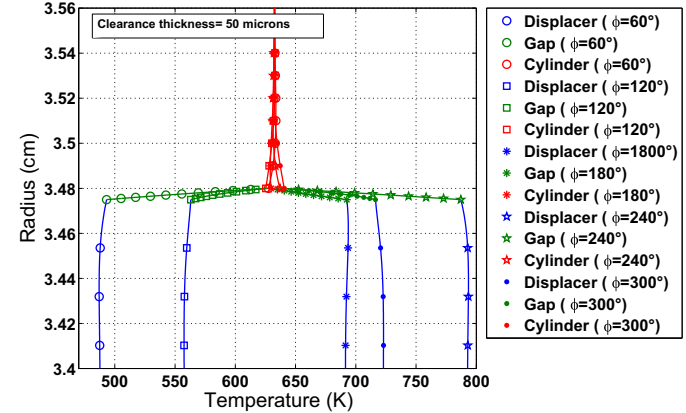


Fig. 11. Radial distribution of the temperature in the displacer, in the gap and in the cylinder for  $z = 0$  and at different crank angles and for two clearance thicknesses.

integration starting from an initial condition defined using the Schmidt analysis [2].

In the simple analysis, heat transfer, pressure drop and effectiveness of the regenerator are taken into account following the scheme presented in Fig. 5.

Input data for the simple analysis are geometric parameters and operating conditions. Instantaneous positions of the piston and the displacer are deduced from a cinematic study of the engine's drive [4]. Instantaneous position and velocity of the displacer are given in Fig. 6.

The pressure difference between the compression and expansion spaces is the sum of pressure drops in the heater, the regenerator and the cooler with respect to the direction of the flow Eq. (27):

$$p_c(t) - p_e(t) = \sum_{i \in \{h,r,k\}} \text{sign}(u_i) \Delta p_i \quad (27)$$

where  $i = h$  for the heater,  $i = k$  for the cooler and  $i = r$  for the regenerator.  $u_i$  is the gas velocity and  $\Delta p_i$  is the pressure drop calculated by:

$$\Delta p_i = \frac{f_i L_i \rho_i u_i^2}{2 D_i} \quad (28)$$

$f_i$  is the friction factor,  $D_i$  is the hydraulic diameter,  $L_i$  is the length and  $\rho_i$  is the gas density.

For the hot and the cold heat exchangers, the friction factor is equal to  $64/Re$  in laminar regions and is calculated using a correlation given by Petukhov [15] in turbulent regions. For the regenerator, the friction factor is calculated using a correlation developed by Tanaka [16]:

- for the heater and the cooler:

$$f_i = 64/Re \quad \text{if } Re_i < 2300 \quad (29)$$

$$f_i = (0.79 \ln(Re_i) - 1.64)^{-2} \quad \text{if } Re_i > 2300 \quad (30)$$

- for the regenerator:

$$f_r = \frac{175}{Re_{r,\max}} + 1.6 \quad (32)$$

These pressure drops and total pressure drop ( $p_c(t) - p_e(t)$ ) are given in Fig. 7 where numerical data are fitted using third order

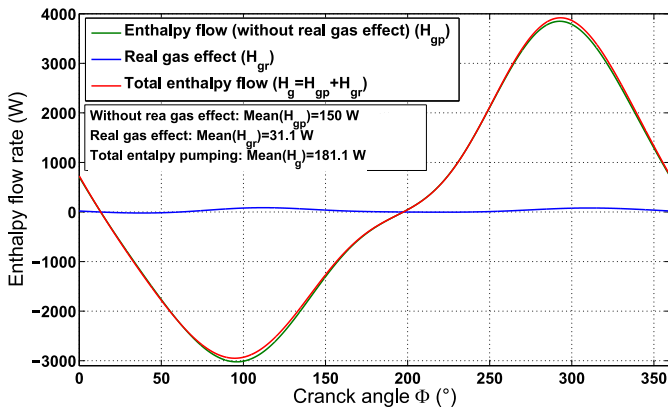


Fig. 12. Enthalpy flow rate through the gap: Enthalpy without real gas effect, real gas effect and total enthalpy flow rate.

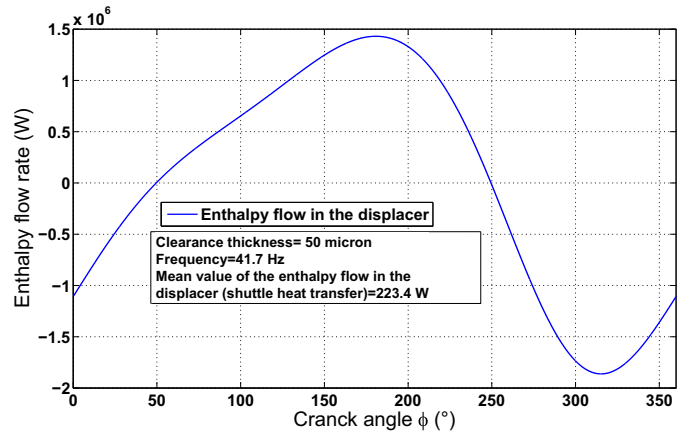


Fig. 13. Enthalpy flow rate in the displacer at  $z = 0$  and its mean value (shuttle heat transfer).

Fourier series. Please notice here that the subscript  $c$  stands for "compression space".

#### 4.2. Velocity and temperature distributions and displacer gap losses

We adopt arbitrary  $z = 0$  and we define the pressure's gradient at this location by a spatially averaged value:  $\partial p(t)/\partial z = (p_c(t) - p_e(t))/L_d$ . For the instantaneous pressure we define it as the arithmetic average between pressures in the compression and expansion spaces  $p(t) = (p_c(t) + p_e(t))/2$ . This is possible without a great error because pressure's spacial variation in the gap is very small compared to its temporal variation.

The instantaneous velocity distribution in the gap between the displacer and the cylinder at this location is then calculated using Eq. (17). In Fig. 8, we show this distribution at different crank angles for a clearance thickness of 50 microns. Notice that it respects boundary conditions. Leakage mass flow rate is then calculated using Equation (20) and shown in Fig. 9 for two clearance thicknesses (50 and 80 microns).

After solving the system of equations constituted by Eqs.(23a)–(23f) for each order, we can give the temperature distributions in the displacer, in the fluid gap and in the cylinder. Gas temperature distribution in the gap is given in Fig. 10 for a clearance thickness of 80 microns and at different crank angles. In Fig. 11 we can find temperature distributions in the three regions (cylinder,

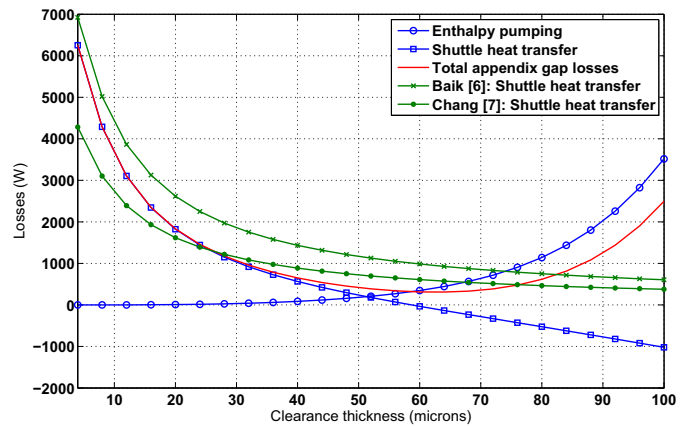


Fig. 14. Effect of the clearance thickness on the shuttle heat transfer and the enthalpy pumping through the displacer's gap.



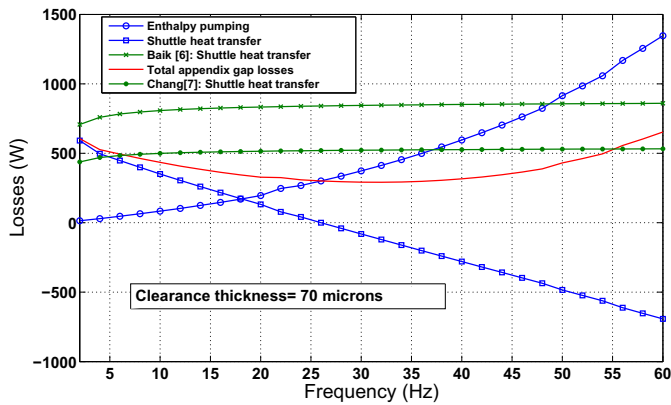


Fig. 15. Variation of the enthalpy pumping, the shuttle heat transfer and the total displacer gap losses as a function of the frequency.

displacer and fluid gap) at different crank angles and for two different clearance thicknesses (50 microns and 80 microns). One can notice in this figure that temperature continuity is respected.

Enthalpy flows in the gap and in the displacer at  $z = 0$  are calculated for a clearance thickness of 50 microns using Eqs. (24) and (25). Fig. 12 shows the enthalpy flow through the gap without real gas effect (the term  $\rho_0 C_{p0} T_f \mu$  only), the effect of real gas (the term  $(1 - T_{f0} \beta_0) p \mu$ ) and their sum which represents total enthalpy pumping through the gap. We note that for this particular case cycle averaged real gas effect represents around 17% of total cycle averaged enthalpy flow pumped through the clearance (31.1 W/180.1 W).

Fig. 13 shows the enthalpy flow in the displacer at the same fixed axial coordinate  $z = 0$  and the cycle averaged value of this enthalpy flow is equal to 223 W which give us total displacer gap losses of 404.5 W.

#### 4.3. Effect of the clearance thickness

Enthalpy pumping and shuttle heat transfer was investigated by varying the clearance thickness and results are given in Fig. 14. Shuttle heat transfer from our model was compared to analytical solutions given by Baik [6] and Chang [7]. We notice that for small clearance thicknesses where the effect of the enthalpy pumping is small, results from Baik's formula is very close to ours.

However, his results become different from ours when the clearance thickness increases. Our results show that shuttle heat transfer decreases more rapidly than predicted by both Baik and Chang and may even be negative. This is due to the effect of the leakage gas flow in the gap which was not taken into account by the former authors. In fact, the displacer exchanges heat with the fluid and returns back a part of the energy pumped through the gap. This makes sense since we notice that gas velocity is always opposite to displacer velocity (Fig. 8) and this is due to the effect of the pressure gradient in the gap that we take into account. So, when clearance thickness rises, enthalpy pumping rises and shuttle heat transfer falls. The consequence of this opposite behavior is that total displacer gap losses present a minimum situated in the studied case between 50 and 70 microns.

#### 4.4. Effect of the engine's speed

It's known that the pressure drop in Stirling engines is more important at high speeds. So, one can expect that engine's speed affects the enthalpy pumping. This fact is confirmed by Fig. 15 where we draw enthalpy pumping as a function of the frequency.

Fig. 15 shows that the enthalpy pumping rises as the frequency rises.

For the shuttle heat transfer, we notice that its behavior is notably different from Baik's and Chang's solutions which predicts that shuttle heat transfer increases slightly up to 15 Hz and is constant for higher frequencies. Our results (Fig. 15) show that shuttle heat transfer decreases when frequency increases. This is due once again to the effect of the leakage flow in the gap which become more important at high speeds.

Consequently, total displacer gap losses present a minimum situated in the studied case between 25 and 35 Hz.

## 5. Conclusion

This study is devoted to "Displacer gap losses" in  $\beta$  and  $\gamma$  Stirling engines. "Displacer gap losses" are the sum of "enthalpy pumping" and the "shuttle heat transfer". Enthalpy pumping is due to the leakage flow between compression and expansion spaces. The shuttle heat transfer is due to the displacer motion between hot and cold regions.

In this work, we develop, in the first part a model for the pulsating fluid flow through the displacer clearance where we take into account the pressure difference between compression and expansion spaces. Gas velocity and temperature distribution in the gap are derived from this analytical model. Conduction heat transfer equations in the cylinder and in the displacer are solved respect to heat and temperature continuity boundary conditions with the gas flowing in the gap.

In the second part, we introduce our model in a global Stirling engine model. GPU-3 is chosen as a study case. In respect of the crank angle, we give gas velocity and temperature distributions in the displacer, in the gap and in the cylinder. Then, we investigate the behavior of the gap losses by varying the clearance thickness and engine's frequency.

For clearance thickness, our results demonstrate that displacer gap losses show a minimum value. For our study case, this minimum is located in the range between 50 and 70 microns. This is a new result compared to former studies.

For frequency, our results demonstrate that displacer gap losses show a minimum value. For our study case, this minimum is located in the range between 25 and 35 Hz. This is also a new result compared to existing studies.

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